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Mon. Not. R. Astron. Soc. **393,** 1370–1376 (2009) doi:10.1111/j.1365-2966.2008.14116.x

Confidence intervals for the correlation between the gamma-ray burst peak energy and the associated supernova peak brightness

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Accepted 2008 October 20. Received 2008 October 20; in original form 2008 May 16

ABSTRACT

A strong correlation between the gamma-ray burster peak energy and the peak luminosity of the associated supernova was discovered by Li for four GRBs. Despite the fact that the formal significance level of the correlation is 0.3 per cent, the smallness of the data set requires careful further evaluation of the result. Subject to the assumption that the data are bivariate Gaussian, a 95 per cent confidence interval of (−0.9972, 0.02) for the correlation is derived. Using data from the literature, it is shown that the distribution of known peak GRB energies is not Gaussian if X-ray flashes are included in the sample. This leads to a proposed alternative to the bivariate Gaussian model, which entails describing the dependence between the two variables by a Gaussian copula. The copula is still characterized by a correlation coefficient. The Bayesian posterior distribution of the correlation coefficient is evaluated using a Markov chain Monte Carlo method. The mean values of the posterior distributions range from −0.33 to about zero, depending on the specifics of the supernova (SN) peak brightness distribution. The implication is that the existing data favour a modest correlation between the GRB peak energy and the SN peak brightness; confidence intervals are very wide and include zero.

Key words: methods: statistical – supernovae: general – gamma-rays: bursts.

1 INTRODUCTION

Many long-duration gamma-ray bursts (GRBs) are accompanied by Type Ic supernovae (SNe) (Woosley & Bloom 2006; Della Vella 2007). In a few cases, the spectroscopic signature of a supernova has been seen in GRB afterglows (e.g. Li 2006 and references therein). More commonly, the non-monotonic afterglow lightcurves of some GRBs have been explained by the presence of a SN component: for example, Ferrero et al. (2006) tabulates estimated luminosities for 13 SNe underlying GRBs with known redshifts.

Li (2006) investigated correlations between the properties of the GRB and those of the associated SNe for four systems in which the connection had been spectroscopically verified. The data are reproduced in Fig. 1. The author found a correlation of $r = -0.997$ between the SN peak bolometric magnitude $M_{\rm bol}$ and the logarithm of the intrinsic peak energy $E_{\rm p}$ of the GRB. It is well known that for a bivariate Gaussian sample of size N,

$$T = \frac{r\sqrt{N-2}}{\sqrt{1-r^2}} \qquad (1)$$

has a t distribution with $(N-2)$ degrees of freedom. The significance level of the correlation is 0.3 per cent according to this t test.

Although the correlation is very significantly non-zero, the data set is quite small and it may be anticipated that any confidence interval for the correlation should be large. This notion is supported by the following argument. Consider again, as above, the null hypothesis of a zero population correlation ($\rho = 0$). Equation (1) can be rewritten as a probability density function (PDF) for r:

$$f(r) = \frac{\Gamma[(N-1)/2]}{\sqrt{\pi}\Gamma[(N-2)/2]}(1-r^2)^{(N-4)/2} \qquad -1 \le r \le 1, \qquad (2)$$

where Γ is the gamma function (e.g. Muirhead 1982). Since the sign of the correlation coefficient is not of interest, the PDF

$$f(|r|) = \frac{2\Gamma[(N-1)/2]}{\sqrt{\pi}\Gamma[(N-2)/2]}(1-r^2)^{(N-4)/2} \qquad 0 \le |r| \le 1 \qquad (3)$$

of the absolute value is more relevant. For a data set of size $N = 4$, this reduces to a uniform distribution – if the true correlation ρ is zero, then *any* observed value of r is *equally* likely. This means, for example, that if the true correlation $\rho = 0$, then the probability of measuring $|r| > 0.95$ is the same as that of measuring $|r| < 0.05$.

The purpose of this paper is to determine confidence intervals for the correlation between $\log E_{\rm p}$ and $M_{\rm bol}$ as shown in Fig. 1. As is demonstrated in Section 2, this is relatively easy if the joint distribution of the two variables is bivariate Gaussian. The assumption of normality is examined in Section 3, where the marginal distributions of $\log E_{\rm p}$ and $M_{\rm bol}$ are studied. This draws on all currently available data on the SN magnitudes and GRB peak energies from the literature. The conclusion is that the joint distribution is *not* Gaussian,

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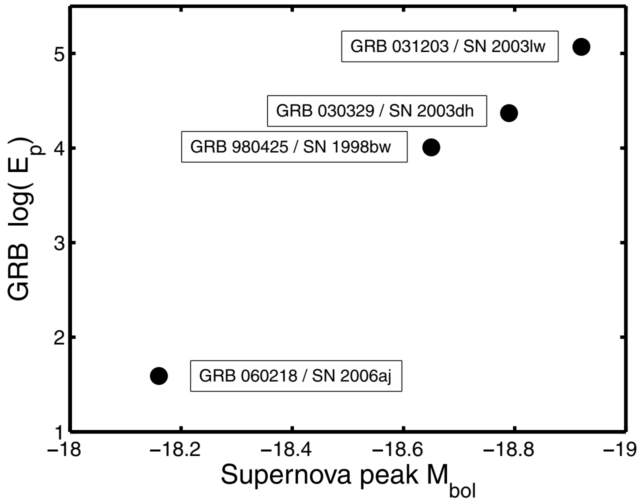

Figure 1. The four data points from Li (2006). Sources of the data are given in table 1 of Li (2006).

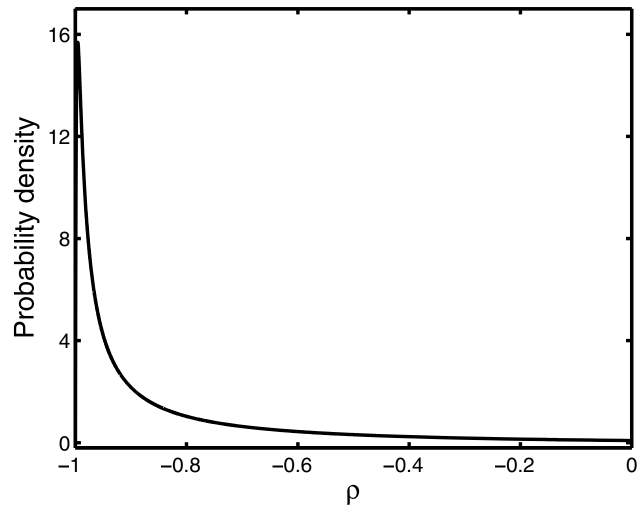

Figure 2. The posterior PDF of the correlation, given four measurements with $r = -0.997$.

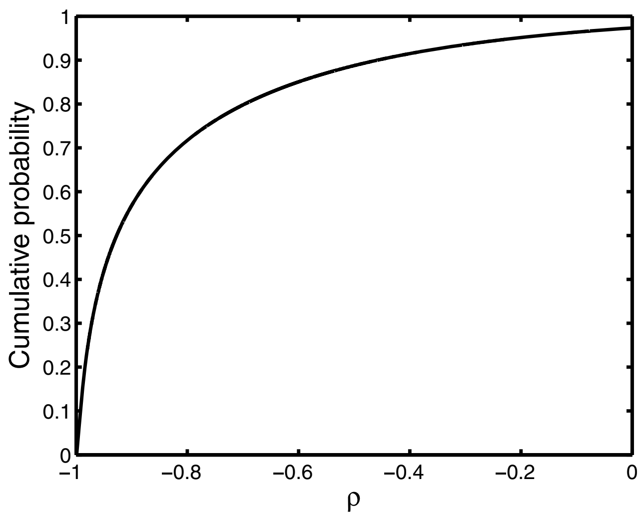

Figure 3. The posterior CDF of the correlation, given four measurements with $r = -0.997$.

since the marginal distributions are not Gaussian. A model for the joint distribution of the two variables is proposed in Section 4. A Markov chain Monte Carlo (MCMC) technique which can then be used to deduce the posterior distribution of ρ, given the data, is described in Section 5. The MCMC results are presented in Section 6, and conclusions are given in Section 7.

The methodology can, in principle, be used to obtain the posterior distribution of the correlation between two quantities with arbitrary marginal distributions, for any sample size.

2 BIVARIATE GAUSSIAN DATA

In order to derive confidence intervals for ρ, the distribution of r for non-zero ρ is required. This is given by

$$f(r) = \frac{(N-2)\Gamma(N-1)}{\sqrt{2\pi}\Gamma(N-1/2)}(1-\rho^2)^{(N-1)/2}(1-\rho r)^{-(2N-3)/2} \times (1-r^2)^{(N-4)/2}\mathcal{H}\left[\frac{1}{2}, \frac{1}{2}; N-\frac{1}{2}; \frac{1}{2}(1+r\rho)\right], \quad (4)$$

where $\mathcal{H}$ is the hypergeometric function (e.g. Muirhead 1982). Since $N = 4$,

$$f(r) = \frac{16\sqrt{2}}{15\pi}(1-\rho^2)^{3/2}(1-\rho r)^{-5/2}\mathcal{H}\left[\frac{1}{2}, \frac{1}{2}; \frac{7}{2}; \frac{1}{2}(1+r\rho)\right]. \quad (5)$$

In order to obtain the posterior distribution of ρ, given the data, use is made of Bayes' theorem:

$$P(\rho|D) = P(D|\rho)P(\rho) \Big/ \int_{-1}^{1} P(D|\rho)P(\rho)\mathrm{d}\rho, \quad (6)$$

where P denotes probability and D the data; $P(a|b)$ indicates the (conditional) distribution of a given b and $P(\rho)$ is the 'prior' distribution of ρ (e.g. Gelman et al. 1995). Throughout this paper, we follow the custom of assuming a uniform prior $P(\rho)$: this implies that, in the absence of any data, all values appear to be equally likely. Setting $P(D|\rho) = f(r) = f(r|\rho)$,

$$\begin{aligned} P(\rho|D) &= f(r|\rho) \Big/ \int_{-1}^{1} f(r|\rho)\mathrm{d}\rho \\ &= f(r = r_*|\rho) \Big/ \int_{-1}^{1} f(r = r_*|\rho)\mathrm{d}\rho, \end{aligned} \quad (7)$$

where $r_* = -0.997$ is the observed value of the correlation coefficient. Equations (5) and (7) can also be compared to the posterior distribution derived by Jeffreys (1983).

The PDF (7) and corresponding cumulative distribution function (CDF),

$$F(x) = \int_{-1}^{x} P(\rho|D)\mathrm{d}\rho, \quad (8)$$

are plotted in Figs 2 and 3. Examination of the CDF in Fig. 3, in particular, shows that there is a small, but not negligible, probability that the true correlation could be quite small in absolute value (e.g. about a 10 per cent chance that $|\rho| < 0.4$). The 95 and 99 per cent confidence intervals for ρ are $(-0.9972,\ 0.02)$ and $(-0.9989,\ 0.43)$, respectively; both of these include zero correlation.

It is interesting to contrast these results with those obtained by the standard approach of using the Fisher statistic, $z = \text{arctanh}\ (r)$. For large bivariate Gaussian samples, z is approximately normally distributed with mean $\eta = \text{arctanh}\ (\rho)$ and variance $1/(N-3)$. In

Table 1. A comparison of SN bolometric peak magnitudes with the luminosities derived by Ferrero et al. (2006). The bolometric magnitudes, from Li (2006), are with respect to the peak magnitude of Supernova 1998bw. The factors k are the peak luminosities, scaled to the luminosity of Supernova 1998b, and the ΔM are the corresponding magnitudes. Scale factors k for Supernova 2006aj were calculated using two different models: both R-band results from Ferrero et al. (2006) are shown in the table.

GRB/SN	$\Delta M_{\rm bol}$	Luminosity ratio k	ΔM
980425/1998bw	0	1	0
030329/2003dh	−0.14	1.50	−0.44
031203/2003lw	−0.27	1.28	−0.27
060218/2006aj	+0.49	0.62, 0.74	0.33, 0.52

the present case, $z = -3.25$ and a 95 per cent confidence interval of $(-5.21, -1.29)$ for η follows. Since $\rho = \tanh(\eta)$, this translates into the approximate interval $(-0.9999, -0.86)$ for the true correlation. The discrepancy with the above results points to the breakdown of this approximation for small samples.

3 MARGINAL DISTRIBUTIONS

3.1 SN peak magnitudes

Table 1 compares the peak bolometric magnitudes of the four SNe from Li (2006) with magnitudes derived from table 3 of Ferrero et al. (2006). The latter were obtained by fitting Supernova 1998bw template lightcurves to the optical observations of the GRB afterglows/SN lightcurves. There is good agreement between the two sets of magnitudes, except for the case of Supernova 2003dh. Study of the literature (see Bloom et al. 2004; Deng et al. 2005 and references therein and Kann, Klose & Zeh 2006) reveals that there are differences in the Supernova 2003dh host galaxy extinction used by different authors. In particular, Deng et al. (2005) (on which the result in Li 2006 is based) assume zero host galaxy extinction, whereas Kann et al. (2006) (as used by Ferrero et al. 2006) obtained $A_V = 0.39 \pm 0.15$ mag. This difference seems sufficient to account for the single discrepancy in Table 1.

In view of the above, it appears reasonable to use the set of nine extinction-corrected luminosity ratios (i.e. luminosities measured with respect to that of Supernova 1998bw) from table 3 in Ferrero et al. (2006) to deduce a distribution of peak magnitudes for GRB SNe. Adding $k = 1$ for Supernova 1998bw and $k = 0.68$ for Supernova 2006aj, the mean offset from Supernova 1998bw is −0.11 mag and the spread (standard deviation) is 0.46 mag. If four GRB SNe without extinction corrections are included, the mean and standard deviation change to 0.17 and 0.62 mag, respectively. Ferrero et al. (2006), on the basis of these data, speculated that the width of the GRB SN luminosity function is at least two magnitudes.

In what follows, it will be assumed that the SN peak bolometric magnitude distribution has a mean in the range −18.5 to −18.75 and a standard deviation between 0.45 and 0.6 mag. Because the sample size is too small to reliably deduce the form of the distribution, the standard assumption of Gaussianity will be adopted: quantile–quantile plots (Figs 4 and 5) show this to be quite reasonable.

As a point of interest, the SN peak magnitude–log $E_{\rm p}$ correlations for these data are mentioned in passing: for the nine extinction-corrected cases with well-determined $E_{\rm p}$ (i.e. Amati 2006, table 1), $r = -0.30$; adding the four cases with no extinction correction gives $r = -0.21$ and including the two GRBs with poorly determined $E_{\rm p}$ (Amati 2006, table 2), $r = -0.20$.

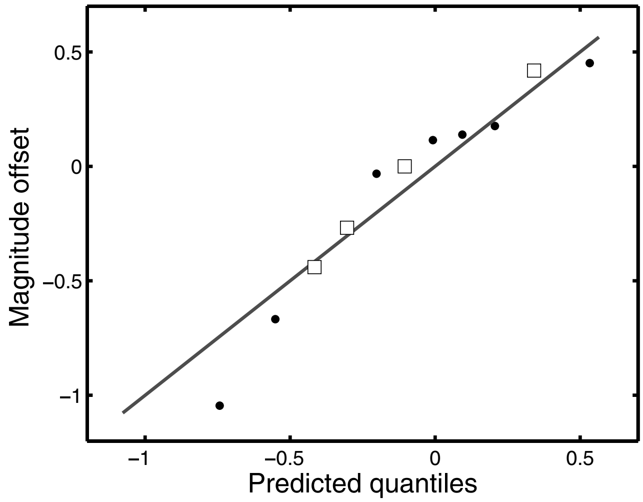

Figure 4. A normal distribution quantile–quantile plot of all the available extinction-corrected SN peak magnitudes, measured with respect to the peak brightness of SN 1998bw. The four spectroscopically identified SNe are marked by squares. The fact that the data show no gross systematic deviations from the line indicates that the magnitudes are roughly Gaussian.

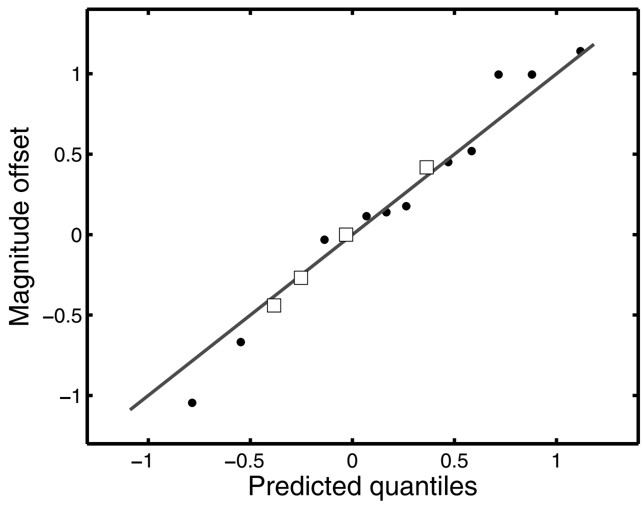

Figure 5. As in Fig. 4, but including four SNe for which extinction corrections have not been made.

3.2 GRB peak energies

Intrinsic peak GRB energies were taken from table 1 of Amati (2006). Both long bursts and X-ray flashes were included. An additional 12 values were collected from Rossi et al. (2008), and GRB 060218 was added from Li (2006); this gave 56 data points in total. There are discrepancies in the values for GRB 030329 [79 keV in Li (2006) and 100 keV in Amati (2006)] and GRB 031203 [159 keV in Li (2006) and 158 keV in table 2 of Amati (2006)]. In both cases, values from Li (2006) are used for consistency with his/her analysis.

Aside from the three extremely sub-energetic sources XRF 020903 (3.37 keV), GRB 060218 (4.9 keV) and GRB 050416A (25.1 keV), the logarithmic data can be well represented by a

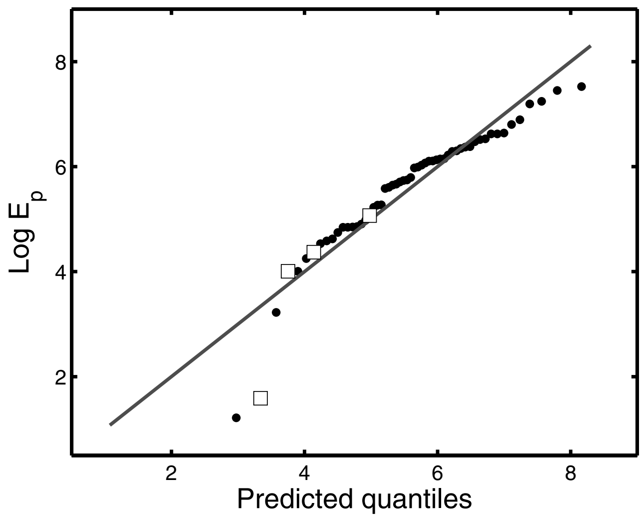

Figure 6. A normal distribution quantile–quantile plot of the available GRB peak intrinsic energies. The four GRB, which are the focus of this paper, are marked by squares. There are 56 data points. Systematic deviations of the extreme energy data points from the line suggest that the log E_{p} are not Gaussian.

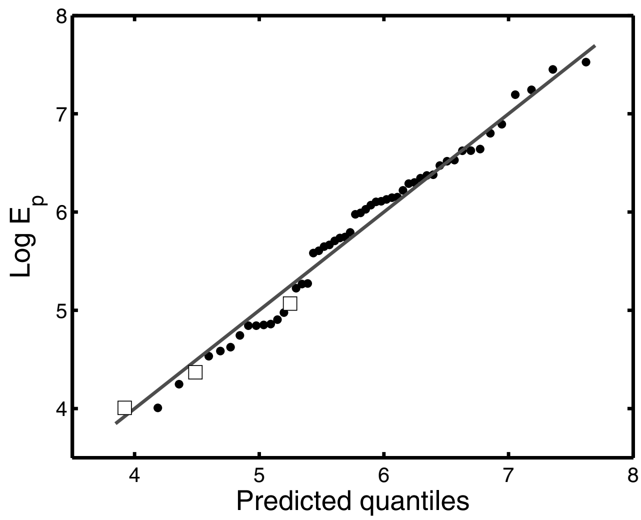

Figure 7. As in Fig. 6, but excluding the three faintest sources. The small scatter around the line indicates that these data are roughly normally distributed.

Gaussian distribution with mean 5.79 and standard deviation 0.88 (Figs 6 and 7). If only the two lowest energy objects are excluded, these numbers change to 5.74 and 0.94, respectively. However, if the bulk of the data are treated as Gaussian, then the energies of both XRF 020903 and GRB 060218 are extremely low-probability events ($p < 10^{-5}$). Given that both of these GRBs had associated SNe (e.g. Ferrero et al. 2006), it is clearly inappropriate to describe the data in terms of a Gaussian defined by the higher energy sources. It is also not acceptable to model the entire distribution by a Gaussian, as Fig. 6 clearly shows. The full empirical distribution will therefore be used in what follows.

Three caveats should be mentioned. The first is the fact that not all long-duration GRBs necessarily have associated SNe (see e.g. Della Valle et al. 2006; Fynbo et al. 2006; Gal-Yam et al. 2006, and the discussions in Modjaz et al. 2008 and Dado et al. 2008). This may mean that the subset of GRBs, which do have SNe, may have a different distribution of peak intrinsic energies. Secondly, it has been suggested that low-energy GRBs are physically distinct from their higher-energy counterparts [see Liang et al. (2007) for a recent discussion of the relevant literature]. Given the current state of knowledge, it is not possible to say whether this would necessarily invalidate the analysis below. Thirdly, GRBs for which only upper or lower limits on E_{p} are known have not been included in the derived distribution. This can, in principle, be remedied, but will not be done here.

4 JOINT DISTRIBUTION

In a bivariate normal distribution, the interdependence between the components is fully described by the pairwise correlations. However, this is not the case for bivariate distributions in general. Instead, the interdependence is generally described by a distribution function known as a copula (e.g. Clemen & Reilly 1999; Genest & Favre 2007). The joint distribution $H(x, y)$ of x and y is then given by

$$H(x, y) = C[F_x(x), F_y(y)]\,, \tag{9}$$

where C is the copula and F_x, F_y are the marginal distribution functions of x and y, respectively. A wide range of copula families are available. In the present context, where the primary interest is in the correlation between two variables, the Gaussian copula

$$C(u, v) = \Phi_{U,V,\rho}[\Phi^{-1}(u), \Phi^{-1}(v)] \tag{10}$$

seems a good choice. In (10), Φ^{-1} is the inverse of the standard normal CDF, and $\Phi_{U,V,\rho}$ is the bivariate normal CDF with covariance matrix

$$\mathbf{R} = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}\,, \tag{11}$$

and $u = F_x(x)$, $v = F_y(y)$. The joint PDF corresponding to (9)–(11) is

$$f(x, y) = f_x(x) f_y(y) |\mathbf{R}|^{-1/2} \exp - \left[\frac{1}{2}\mathbf{z}'(\mathbf{R}^{-1} - \mathbf{I})\mathbf{z}\right] \tag{12}$$

(e.g. Clemen & Reilly 1999), where $\mathbf{I}$ is the identity matrix and

$$\mathbf{z} = \begin{bmatrix} \Phi^{-1}[F_x(x)] \\ \Phi^{-1}[F_y(y)] \end{bmatrix}\,. \tag{13}$$

The marginal PDFs of x and y are denoted by f_x and f_y, respectively.

For a specified correlation coefficient ρ, equation (12) can now be used to find the joint probability P of a set of measurement pairs ($\log E_{\mathrm{p},j}, M_j$) ($j = 1, 2, \ldots, N$) of GRB intrinsic peak energies and associated SN peak bolometric magnitudes:

$$\begin{aligned} P &= |R|^{-N/2} \prod_{j=1}^{N} f_{\mathrm{E}}(\log E_{\mathrm{p},j}) f_{\mathrm{M}}(M_j) \exp - \left[\frac{1}{2}\mathbf{z}_j' \left(\mathbf{R}^{-1} - \mathbf{I}\right) \mathbf{z}_j\right] \\ \mathbf{z}_j &= \begin{bmatrix} \Phi^{-1}[F_{\mathrm{E}}\left(\log E_{\mathrm{p},j}\right)] \\ \Phi^{-1}[F_{\mathrm{M}}(M_j)] \end{bmatrix}. \end{aligned} \tag{14}$$

In (14), f_{E} and f_{M} are the marginal PDFs of the peak GRB (log) energy and the peak SN magnitude, respectively, as obtained in the previous section of the paper. The corresponding CDFs are F_{E} and F_{M}.

Since the distribution of SN magnitudes M_j is assumed to be Gaussian, it follows that

$$\Phi^{-1}[F_{\mathrm{M}}(M_j)] = \frac{(M_j - \overline{M})}{\sigma_{\mathrm{M}}}, \tag{15}$$

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where $\overline{M}$ and σ_{M} are the mean and standard deviation of the magnitudes. The right-hand side of (15) is the normal 'score' of M_j.

The CDF of the GRB peak energies can be estimated by the *empirical* CDF

$$\widehat{F}_{\mathrm{E}}(\log E_{\mathrm{p},j}) = \frac{1}{M}\sum_{k=1}^{M} \#(E_{\mathrm{p},k} \le E_{\mathrm{p},j}) \tag{16}$$

(i.e. the fraction of the sample which is smaller than, or equal to, $E_{\mathrm{p},j}$). The size of the available sample of GRB peak energies is denoted by M ($M = 56$ in Section 2.2).

5 THE MARKOV CHAIN MONTE CARLO METHOD

Only an abbreviated description of MCMC methodology follows – the interested reader is referred to e.g. Gelman et al. (1995) for detail. Astronomical applications can be found in e.g. Ford (2005) and Croll (2006). In essence, MCMC is a Bayesian technique which can be used to obtain marginal posterior distributions of parameters of interest, given a data set and a form (which may be quite complicated) for the joint posterior distribution of the entire parameter set. Most of the discussion below will refer to the bivariate Gaussian as an example distribution.

Denote by $\boldsymbol{\theta} = \{\theta_1, \theta_2, \ldots, \theta_T\}$ the set of parameters characterizing the full data distribution. In the case of bivariate normal data, for example, $T = 5$ – two mean values (μ_x, μ_y), two variances (σ_x^2, σ_y^2) and the correlation ρ. The joint posterior distribution of the components of $\boldsymbol{\theta}$ is given by Bayes' theorem as

$$P(\boldsymbol{\theta}\,|\,D) \propto P(\boldsymbol{\theta})P(D\,|\,\boldsymbol{\theta}) \,. \tag{17}$$

The term $P(\boldsymbol{\theta})$ is the prior distribution of $\boldsymbol{\theta}$.

For the bivariate normal, standard assumptions are 'flat' (i.e. constant) priors for the means and for ρ, while $P(\sigma^2) \propto \sigma^{-2}$ in the case of the variances. The factor $P(D\,|\,\boldsymbol{\theta})$ is the likelihood of the data, given by

$$P(D\,|\,\boldsymbol{\theta}) \propto |\Sigma|^{-N/2} \exp\left\{ -\frac{1}{2}\sum_{j=1}^{N} [x_j - \mu_x \quad y_j - \mu_y] \right.$$
$$\left. \times\, \Sigma^{-1}[x_j - \mu_x \quad y_j - \mu_y]' \right\}, \tag{18}$$

where the data consist of the pairs of measurements (x_j, y_j) ($j = 1, 2, \ldots, N$). The covariance matrix in this case is

$$\Sigma = \begin{bmatrix} \sigma_x^2 & \rho\sigma_x\sigma_y \\ \rho\sigma_x\sigma_y & \sigma_y^2 \end{bmatrix} \,. \tag{19}$$

The marginal distributions of the individual θ_j can be obtained from (17) in at least two ways: conceptually, the simplest is to integrate with respect to the remainder of $T - 1$ parameters. In the case of the bivariate normal, this would require four-dimensional integrals, which would be tedious to evaluate numerically. The alternative we pursue is the MCMC method. In essence, candidate values of the θ_j are drawn randomly from 'proposal' distributions and the candidates are then either accepted or rejected. The acceptance probability of the *j*th draw is

$$p = \min\left[1, \frac{P\left(D\,|\,\boldsymbol{\theta}_{(j)}\right)}{P\left(D\,|\,\boldsymbol{\theta}_{(j-1)}\right)}\right] \,. \tag{20}$$

Thus, for example, $\boldsymbol{\theta}_{(j)}$ will definitely be accepted if its posterior probability is larger than the posterior probability of the previous draw.

An important ingredient of the MCMC method is the specification of the proposal distribution from which candidate members $\boldsymbol{\theta}$ are drawn. As a test, the method was applied to the bivariate normal distribution. For the means and standard deviations, the 'Metropolis sampler' (Metropolis et al. 1953) based on normal distributions was used: for the *j*th draw of the *k*th parameter

$$\theta_{k,(j)} \sim N\left(\theta_{k,(j-1)}, S_k^2\right), \tag{21}$$

i.e. $\theta_{k,(j)}$ is drawn from a Gaussian centred on the previously accepted value. The scale S_k of the Gaussian is adjusted to give quick convergence of the distributions [for details, see e.g. Gelman et al. (1995), Ford (2005) or Croll (2006)]. An 'independence sampler' (Tierney 1994) seemed to work best for ρ: each candidate is drawn independently of previous draws from a uniform distribution on the interval $[-1, 1]$. The MCMC results agreed well with the known forms (e.g. Gelman et al. 1995) for the marginal posterior distributions of the five parameters of the bivariate normal distribution.

6 RESULTS

The MCMC theory described in Section 5 is now applied to the bivariate distribution (14), with the data D being the four $M_{\mathrm{bol}} - \log E_{\mathrm{p}}$ pairs studied by Li (2006). Substitution of (14) into (20) shows that the acceptance probability is independent of the two marginal distributions, and the vector of parameters $\boldsymbol{\theta}$ reduces to the scalar ρ:

$$p = \min\left\{1, \left[\frac{|\mathbf{R}_{(j-1)}|}{|\mathbf{R}_{(j)}|}\right]^{N/2} \exp -\left[\frac{1}{2}\sum_k \mathbf{z}_k' \left(\mathbf{R}_{(j)}^{-1} - \mathbf{R}_{(j-1)}^{-1}\right)\mathbf{z}_k\right]\right\}. \tag{22}$$

Markov chains were generated for a range of parameter values for the Gaussian marginal distributions of the SN brightnesses (see Section 3.1). All the 99 per cent confidence intervals for the correlation comfortably included zero (Table 2). The implication is that the information contained in the marginal distributions does not support the strong correlation between the GRB peak energy and the SN brightnesses seen in the four data points from Li (2006).

Example posterior probability densities are shown in Fig. 8. Each of these is based on 200 000 MCMC-generated values.

7 CONCLUSIONS

The main findings of this paper are as follows.

(i) Bivariate Gaussian data sets have the remarkable property that if the x and y variables are independent, then for a sample size $N = 4$, all observed correlations are equally likely. This means that *any* measured correlation for such small data sets should be taken with a pinch of salt.

(ii) The large-sample Fisher's arctanh transformation cannot be used to obtain confidence intervals for ρ for samples as small as $N = 4$ – it returns intervals which are far too narrow.

(iii) Although the correlation $r = -0.997$ found by Li (2006) is extremely strong, the confidence intervals for the population correlation ρ are indeed very wide, as shown in Section 2 (assuming a bivariate Gaussian distribution).

(iv) Although the peak magnitudes of SNe associated with GRB may have a Gaussian distribution (Figs 4 and 5), the same cannot be said for the (log) GRB peak energies (Fig. 6). This implies that these two variables cannot have a joint Gaussian distribution.

(v) A more appropriate description of the joint distribution of the SN peak magnitude and $\log E_{\mathrm{p}}$ is provided by equation (14);

Table 2. Summaries of the posterior distributions of the correlation coefficient ρ, for various parameters of the Gaussian describing the marginal distribution of the GRB-associated SNe. For each model, 10 parallel MCMC chains of length 25 500 each were generated. The first 500 values were discarded to eliminate contamination by start-up values.

μ_M	σ_M	$\widehat{\rho}$	95 per cent confidence interval	99 per cent confidence interval
−18.75	0.45	−0.33	−0.79, 0.33	−0.85, 0.48
−18.75	0.60	−0.27	−0.77, 0.40	−0.83, 0.54
−18.65	0.45	−0.18	−0.70, 0.45	−0.78, 0.58
−18.65	0.60	−0.15	−0.69, 0.50	−0.78, 0.62
−18.50	0.45	0.04	−0.55, 0.60	−0.65, 0.69
−18.50	0.60	0.03	−0.57, 0.61	−0.68, 0.71

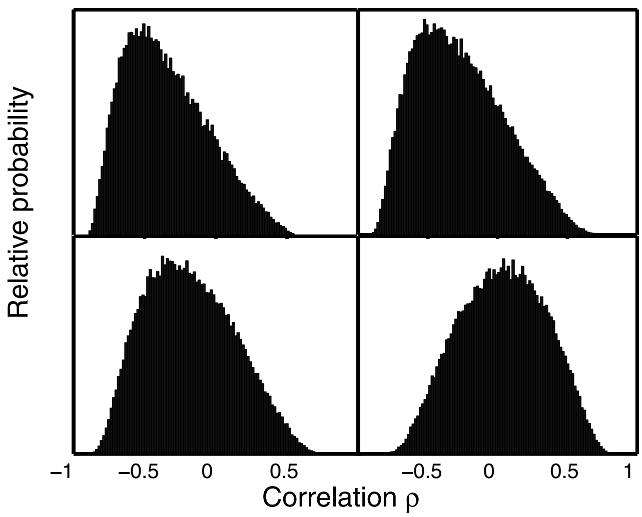

Figure 8. Example posterior distributions of the GRB peak energy–SN peak magnitude correlation for various parameters describing the marginal distribution of the SN peak brightness. Top left panel: ($\mu_M = -18.75$, $\sigma_M = 0.45$). Top right panel: ($\mu_M = -18.75$, $\sigma_M = 0.6$). Bottom left panel: ($\mu_M = -18.65$, $\sigma_M = 0.45$). Bottom right panel: ($\mu_M = -18.5$, $\sigma_M = 0.45$).

the connection between the two variables is still characterized by a correlation, but the two individual distributions need no longer be Gaussian.

(vi) The two marginal distributions deduced in Sections 3.1 and 3.2 were substituted into (14), and a MCMC method was used to produce a posterior distribution of ρ from the four data pairs given by Li (2006). The correlations found are not particularly strong, and the confidence intervals are very wide (Table 2 and Fig. 8).

(vii) The implication of point (vi) is that the two observed marginal distributions, and the four specific data points of Li (2006), do not together imply a strong correlation between the peak SN magnitude and log E_p.

There are a number of points worth further work.

(i) The quoted measurement errors in the $E_{p,k}$ and M_k have not been incorporated into the above analysis. Effectively, the above analysis was therefore performed on the data, which have convolved distributions. Note that the same applies to the calculations in Li (2006). There are some improvements which could be explored. First, the two underlying marginal distributions discussed in Section 3 could be estimated by deconvolution, taking into account the individual measurement error specifications. Secondly, if bivariate normality is assumed, then the example MCMC calculations of Section 5 could be simply extended by adding the measurement error variances to the (log E_p, M) covariance matrices.

Aside from random errors, widely different results have been obtained from the observational material by different authors. To give but two examples, Watson et al. (2004) derive $E_p < 20$ keV for GRB 031203 [as opposed to 159 keV used by Li (2006)], and Kaneko et al. (2007) find $E_p \sim 122$ keV for GRB 980425 [as opposed to 55 keV used by Li (2006)]. Of course, this will also contribute to the uncertainty in ρ.

(ii) In principle, it is also possible to improve on the results in Section 6 by modelling the uncertainties in the two marginal distributions. This could be accomplished by running simultaneously MCMC chains for each of the two marginal distributions, and for the dependence structure. In the case of the marginal distributions, the two data sets of Sections 3.1 and 3.2 would be used, while the Li (2006) data set would be used to obtain the posterior distribution of ρ (as in Section 6). The distribution of the SN magnitudes is Gaussian, and MCMC chains could be used to obtain the posterior distributions of the mean and variance (which were fixed in the Section 6 calculations). As far as the GRB peak energies are concerned, a beta distribution, or perhaps a mixture of normals, could be used. Posterior distributions of the parameters would again follow from MCMC chains.

(iii) An important point regarding Li's (2006) study, which has not been addressed above, is the effect of multiple hypothesis testing. The GRB peak energy–SN magnitude correlation is in fact the strongest of a *number* of pairwise correlations between different variables studied (including the GRB isotropic equivalent energy and the SN kinetic energy.) Clearly, the significance level of the maximum of a number of correlation coefficients is less impressive than that of a single pre-selected correlation coefficient. The p value of 0.3 per cent referred to in Section 1 is therefore overly optimistic.

The methodologies presented in Sections 2, 4, 5 and 6 are not specific to the problem discussed in this paper and could therefore, at least in principle, be used to obtain small sample confidence intervals for correlation coefficients in general.

ACKNOWLEDGMENTS

I am grateful to the referee for supplying a number of pertinent references.

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